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LETTER TO THE EDITOR

Isospectral SUSY potentials

Cao xuan Chuan†

Observatoire de Nice, BP 135, Nice, France

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Abstract. We introduce a new linear transformation in order to construct isospectra potentials in the frame of supersymmetric quantum mechanics. Under this transformation, the adjoint character of the 'ladder operator' is destroyed but with preservation of the energy spectrum leading to some interesting consequences such as the concept of pseudo-partnership of the two components. Connection with the Darboux Theorem as well as with some other recent methods of construction is analysed.

Iso- or pseudoisospectral potentials, whose study is increasing in impetus because of their connections with many fields of quantum mechanics, are defined as members of families of potentials which may depend on a single or several parameters and which yield identical energy spectrum of the Schrödinger equations. They differ among themselves in the behaviour of their phase shift but generally conserve the same formal analytical representation. There is however, no unique approach to construct these families as has been remarked earlier (Luban and Pursey 1986) for instance in the inequivalence between the Darboux construction (1982) and the Abraham–Mose method (1981) which is based on the use of the Gel'fand–Levitan equation or on the Marchenko equation (Pursey 1986).

Isospectral SUSY potentials which are directly related to supersymmetric quantum mechanics on the other hand are expected to present new problems because in simple isospectral potentials, the mathematical approach is based on a single (one-component) Schrödinger equation while in SUSY isospectral potentials, we have to deal simultaneously with at least a couple of Schrödinger equations in the frame of the two-components theory. This may lead to new features which are not predicted in a one-component approach.

In the present letter, we shall explore this second point of view, in relying on two types of transformation, the C and its modified form $\hat{C}$ transformations. It will be seen that the effect of the first kind of transformation is to preserve supersymmetry or, more precisely, to preserve the adjoint character of the 'ladder operator' but altering the energy spectrum. In the second one, however, this adjointness is lost but with preservation of the energy spectrum hence providing a convenient point of view to approach the construction of families of isospectral potentials.

The C transformation. We use the same notations as previously in which ϕ_1, ϕ_2 are the bosonic and fermionic components, $dv/dx = v'$, $v(x)$ is the superpotential and the ladder operators $A^{\pm}$ are defined by $A^{\pm} = \pm d/dx + v'$ so that (Cao 1990a):

$$A^{+}\phi_1 = \sqrt{2E}\,\phi_2 \qquad A^{-}\phi_2 = \sqrt{2E}\,\phi_1 \tag{1}$$

† Permanent address: 01 Parvis du Breuil, 92160 Antony, France.

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E being a member of the energy spectrum $\{E_n\}$ n:0, 1, 2.... Consider now a new representation $(\bar{\phi}_1, \bar{\phi}_2) = \bar{\phi}$ such that

$$\bar{\phi} = C^{-1}\phi \qquad \text{in which} \qquad C = \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \tag{2}$$

$c(x)$ being an arbitrary function of x. Under this transformation system (1) becomes

$$\bar{A}^{+}\bar{\phi}_1 = \sqrt{2E}\ c^2\bar{\phi}_2 \qquad \bar{A}^{-}\bar{\phi}_2 = \sqrt{2E}\frac{1}{c^2}\bar{\phi}_1 \tag{3}$$

in which the new ladder operator are defined by $\bar{A}^{\pm} = \pm \mathrm{d}/\mathrm{d}x + \bar{v}'$, $\bar{v}(x)$ being the new superpotential in the $(\bar{\phi})$ representation, $\bar{v}' = v' + c'/c$.

Therefore, the adjoint character of the ladder operator is preserved (implying conservation of supersymmetry) but with alteration of the energy spectrum i.e. $\{\bar{E}_n\} \neq \{E_n\}$.

The $\hat{C}$ transformation. Consider now a second representation $\bar{\bar{\phi}} = (\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_2)$ defined by

$$\bar{\bar{\phi}} = \hat{C}^{-1}\phi \tag{4}$$

$\hat{C} = c(x)I$, I being the unit matrix. Under (4) the system (1) can be written as:

$$\hat{A}_{\alpha}^{+}\bar{\bar{\phi}}_1 = \sqrt{2E}\ \bar{\bar{\phi}}_2 \qquad \hat{A}_{\beta}^{-}\bar{\bar{\phi}}_2 = \sqrt{2E}\ \bar{\bar{\phi}}_1 \tag{5}$$

in which the new ladder operators are:

$$\hat{A}_{\alpha}^{+} = \frac{\mathrm{d}}{\mathrm{d}x} + v' + \frac{c'}{c} \qquad \hat{A}_{\beta}^{-} = -\frac{\mathrm{d}}{\mathrm{d}x} + v' - \frac{c'}{c} \tag{6}$$

which obviously are not adjoint, the indices α, β serving for labels but the energy spectrum is preserved. This also means that there are now two new superpotentials

$$\hat{v}_{\alpha}' = v' + \frac{c'}{c} \tag{7a}$$

$$\hat{v}_{\beta}' = v' - \frac{c'}{c} \tag{7b}$$

each of them generating two new adjoint ladder operators

$$\hat{A}_{\alpha}^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \hat{v}_{\alpha}' \qquad \hat{A}_{\beta}^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}x} + \hat{v}_{\beta}' \tag{8}$$

with two pairs of SUSY partners $(\bar{\bar{\phi}}_1, \bar{\bar{\phi}}_{1,s})(\bar{\bar{\phi}}_2, \bar{\bar{\phi}}_{2,s})$ with

$$\hat{A}_{\alpha}^{+}\bar{\bar{\phi}}_1 = \sqrt{2E}\ \bar{\bar{\phi}}_{1,s} \qquad \hat{A}_{\alpha}^{-}\bar{\bar{\phi}}_{1,s} = \sqrt{2E}\ \bar{\bar{\phi}}_1 \tag{9a}$$

$$\hat{A}_{\beta}^{+}\bar{\bar{\phi}}_{2,s} = \sqrt{2E}\ \bar{\bar{\phi}}_2 \qquad \hat{A}_{\beta}^{-}\bar{\bar{\phi}}_2 = \sqrt{2E}\ \bar{\bar{\phi}}_{2,s}. \tag{9b}$$

Isospectral potentials. Consider for instance system (9a) with $\hat{v}' = v' + c'/c$ (the index α is omitted for simplicity). Form as usual the two Hamiltonians $2H_+ = \hat{A}^-\hat{A}^+$, $2H_- = \hat{A}^+\hat{A}^-$, with two corresponding potentials $\hat{V}_{\pm} = \hat{v}'^2 \pm \hat{v}''$. We may subject $c(x)$ to the conndition:

$$\hat{V}_+ = \hat{v}'^2 + \hat{v}'' = v'^2 + v''. \tag{10}$$

$c(x)$ must be a solution of the following differential equation:

$$c''+2v'c'=0. \tag{11}$$

We find:

$$c(x)=e^{\lambda_1}\int_{-\infty}^{x} e^{-2v(x)}\,dx+\lambda_2 \tag{12}$$

λ_1, λ_2 being constants of integration. We shall discuss the following two cases.

Case 1. If $\{E_n\}$ contains a non-degenerate zero-energy ground state ($E_0=0$) which implies existence of a normalizable wavefunction ($\phi_{1,0}\simeq e^{-v}$ so that the quantity $d\log c/dx$ can be writtten as

$$\frac{d}{dx}\log c(x)=\phi_{1,0}^2\left(\int_{-\infty}^{2}\phi_{1,0}^2\,dx+\lambda\right)^{-1} \tag{13}$$

in which $\lambda=\lambda_l\,e^{-\lambda_1}$. In this sense we may conclude that the new superpotential $\hat{v}(x)=v(x)+\log c(x)$ generates a one-parameter family of isospectral potentials corresponding to $\{E_n\}$. The partner of $\hat{V}_+$ is

$$\hat{V}_-=\hat{v}'^2-\hat{v}''=V_--2\frac{d^2}{dx^2}\log C(x). \tag{14}$$

The new eigenfunction related to the ground state is, A being a constant of normalization,

$$\bar{\bar{\phi}}_{1,0}=\frac{A}{c(x)}\phi_{1,0} \tag{15}$$

$$A=e^{\lambda_1/2}\sqrt{\lambda(\lambda+1)}. \tag{16}$$

Therefore $\bar{\bar{\phi}}_{1,0}$ is defined only when $\lambda>0$ or $\lambda<-1$. Symmetry is broken if $\lambda=0$ (corresponding to the Pursey potential) and when $\lambda=-1$ (corresponding to the Abraham–Mose potential) because of deletion of the ground state making the spectra of $\hat{V}_+$, $\hat{V}_-$ completely degenerate. These conclusions agree exactly with those pointed out previously by Khare and Sukhatme (1988) who used a slightly different approach.

Case 2. Note first that relation (15) results from the special structure of $\phi_{1,0}$ ($\phi_{1,0}\simeq e^{-v(x)}$). However, if we assume now that the energy of the ground state is $E_0>0$, the state E_0 being either non-degenerate or degenerate, the first case corresponding to an even superpotential ($v(x)=v(-x)$), the second one to an odd potential ($v(x)=-v(-x)$). For even potentials, $\int_{-a}^{+\infty}e^{-2v(x)}\,dx=\beta$; choosing $\lambda>\beta$, we see that the quantity $\log c(x)$ is always defined. On the other hand, as ϕ_{1,E_0} is assumed to be normalizable, the new ground-state wavefunction from (4)

$$\bar{\bar{\phi}}_{1,E_0}=\frac{\bar{A}}{c(x)}\phi_{1,E_0} \tag{17}$$

will also be normalizable. The normalization condition here depends on the analytical form of ϕ_{1,E_0}.

For odd potentials, the integral in (15) is divergent so that $c(x)$ cannot be defined with this approach.

Connection with the Darboux theorem. We can also solve equation (11) by setting†

$$c(x) = \gamma(x)\,\mathrm{e}^{-v(z)} \tag{18}$$

where $\gamma(x)$ is now an unknown function of x. With (11) it can be verified that $\gamma(x)$ must satisfy the equation:

$$-\gamma'' + V_+\gamma = 0 \tag{19}$$

which means that this function can be identified with $\phi_{2,0}$ of system (1). If $E_0 > 0$, $\phi_{2,0}$ is not normalizable but it is always possible to construct it such that it is nodeless for any finite x. For example if ψ_0 is a particular solution of (19), the general form of $\phi_{2,0}$ is (Sukumar 1985a, b):

$$\phi_{2,0} = \psi_0\left[\lambda + \int_{-\infty}^{x} \psi_0^{-2}\,\mathrm{d}x\right]$$

λ being a parameter. From (7a) and (18) we may conclude that the new superpotential $\hat{v}(x)$,

$$\hat{v}(x) = \log\phi_{2,0}(x) + \text{constant} \tag{20}$$

is definite everywhere and can be used to construct a second one-parameter family of isospectral potentials. In our representation ($\bar{\bar{\phi}}_1$, $\bar{\bar{\phi}}_{1,s}$), the corresponding Schrödinger equations are

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+\right]\bar{\bar{\phi}}_{1,s} = 2E\bar{\bar{\phi}}_{1,s} \tag{21a}$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+ - 2\frac{\mathrm{d}^2}{\mathrm{d}x^2}\log\phi_{2,0}\right]\bar{\bar{\phi}}_1 = 2E\bar{\bar{\phi}}_1. \tag{21b}$$

Equation (21a) reflects the fact that the spectra corresponding to $\bar{\bar{\phi}}_{1,s}$ and ϕ_2 are identical ($E = \{E_0, E_1, \ldots\}$). Equation (21b) means that for $\bar{\bar{\phi}}_1$, we have the same spectrum but with an additional zero-energy state with eigenfunction $\bar{\bar{\phi}}_{1,0}$. In order to prove that $\bar{\bar{\phi}}_{1,0}$ is normalizable we note from (4) that $\bar{\bar{\phi}}_{1,0} = (1/c(x))\phi_{1,0}$. From (18) and noting that $\phi_{1,0} \simeq \bar{A}\,\mathrm{e}^{-v}$, we obtain

$$\bar{\bar{\phi}}_{1,0} = \frac{\bar{A}}{\phi_{2,0}}\,\phi_{1,0}. \tag{22}$$

Although $\phi_{2,0}$ is not normalizable, as it is assumed to be nodeless its inverse will be finite everywhere and $\bar{\bar{\phi}}_{1,0}$ is normalizable. Furthermore we may write from (9a)

$$\begin{aligned}\bar{\bar{\phi}}_{1,E} &= \frac{1}{\sqrt{2E}}\hat{A}^-\bar{\bar{\phi}}_{1,s,E} = \frac{1}{\sqrt{2E}}\left[-\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\phi_{2,0}'}{\phi_{2,0}}\right]\bar{\bar{\phi}}_{1,S,E}\\ &= \frac{1}{\sqrt{2E}}\frac{1}{\phi_{2,0}}W(\bar{\bar{\phi}}_{1,S,E}, \phi_{2,0})\end{aligned}$$

W denoting the usual symbol for the Wronskian.

Therefore, with the use of the $\hat{C}$ transformation and the relative simple reasoning above, we have just obtained an alternative proof of the Darboux theorem (1882) from the point of view of supersymmetric quantum mechanics. Incidentally, we may note

† $v(x)$ may now be odd in the sense $v(x) = -v(-x)$ (symmetry breaking).

the close similarity between the roles of the functions $\bar{\bar{\phi}}_1(x)$, $c(x)$ which do have a precise physical interpretation and the functions $u(x)$, $-\theta(x)$ which are pure mathematical entities in Darboux's work. Note also that the result (20) remains valid regardless of the parity of the superpotential $v(x)$.

Example 1. We take the case $v(x)=\frac{1}{2}x^2$ (the oscillator problem) for which the particular solution ψ_0 is well known:

$$\psi_0=\mathrm{e}^{x^2/2}\sum_{m=0}^{\infty} a_m x^{2m}$$

where a_m are positive coefficients making ψ_0 non-normalizable. The general form from $\phi_{2,0}(\lambda, x)$ is:

$$\phi_{2,0}(\lambda, x)=\psi_0\left(\lambda+\int_{-\infty}^{x}\psi_0^{-2}\,\mathrm{d}x\right)$$

(where λ is a parameter) and is nodeless. Therefore with the above approaches we obtain the following types of superpotentials:

(i) $v(x)=\frac{1}{2}x^2$

(ii) $\hat{v}(x)=\frac{1}{2}x^2+\log[\mathrm{e}^{\lambda_1}\int_{-\infty}^{x}\mathrm{e}^{-x^2}\,\mathrm{d}x+\lambda_2]$

(iii) $\hat{v}(x)=\log[\phi_{2,0}(\lambda, x)]$.

It can be recognized that (i) corresponds to the usual harmonic oscillator while the two other cases give rise to anharmonic oscillators; they have the same energy spectrum with an additional zero-energy state for the third one.

Remark 1. Recall that the scattering matrices corresponding to the partner potentials are related by

$$\hat{S}_-(k)=\frac{\hat{v}'_+-\mathrm{i}k}{\hat{v}'_++\mathrm{i}k}\hat{S}_+(k) \qquad k=\sqrt{E-v_+'^2}$$

where $\hat{S}(k)=\mathrm{e}^{\mathrm{i}\delta(k)}$, $\hat{v}'_\pm=v'(x\to\pm\infty)$. If from (12) and (20) we have

$$\lim_{x\to\infty}\frac{\mathrm{d}}{\mathrm{d}x}\log c(x)=0$$

then $\hat{v}'_\pm=v'_\pm$. This means that $\hat{S}(k)=S(k)$ and the phase shift will remain invariant, making these potentials strictly isospectral.

Remark 2. In case (ii), if we set $\mathrm{e}^{\lambda_1}=2/\sqrt{\pi}$, $\lambda_2=0$, then the quantity $\mathrm{d}/\mathrm{d}x\log c(x)$ can be written as

$$\frac{c'}{c}=-\frac{2}{\sqrt{\pi}}\frac{\mathrm{e}^{-x^2}}{\operatorname{erf} c(x)}$$

where

$$\operatorname{erf} c(x)=\frac{2}{\sqrt{\pi}}\int_x^{\infty}\mathrm{e}^{-x^2}\,\mathrm{d}x.$$

Formally it has the same analytical expression as the Gel'fand-Levitan kernel $K(x,x)$ in the Abraham-Moses construction (1980) with addition of a ground state. Note that in spite of this formal similarity, these two approaches cannot be equivalent because $K(x,x)$ is considered as an integral operator (the U transformation) while the quantity $\mathrm{d}\log c(x)/\mathrm{d}x$ here results from a linear transformation (the $\hat{C}$ transformation).

Remark 3. It is also interesting to point out some other use of relation (18). For instance, if in state $\phi_{2,0}$, we take the case $\gamma(x)=\phi_{2,E_0}$, E_0 being the first bound state, then equation (21*a*) can be written as

$$H\bar{\bar{\phi}}_{1,s,E_0}=E_0\bar{\bar{\phi}}_{1,s,E_0} \qquad 2H=\hat{A}^+\hat{A}^- - E_0.$$

One may recognize here the basic idea leading to the well known technique of factorization (Andrianov *et al* 1984, Sukumar 1985a).

Iteration. The new superpotential $\hat{v}(x)$ can in principle be used again to start another similar construction with the relation

$$c^{(p)}_{(x)}=\int_{-\infty}^{x} e^{-2v(p-1)}(x)\,dx+\lambda^{(p)}$$

in which (p) denotes iteration of order p,

$$\hat{v}^{(p-1)}(x)=v(x)+\log\left|\prod_{i=1}^{p-1}c^{(i)}(x)\right|. \tag{23}$$

For instance, with two repeated operations we obtain, after some simple algebra:

$$C^{(1)}C^{(2)}=MC^{(1)}+N \tag{24}$$

in which

$$M=\frac{1}{\lambda_2^{(v)}}e^{\lambda_1^{(2)}-\lambda_1^{(1)}} \qquad N=-e^{\lambda_1^{(2)}-\lambda_1^{(1)}}.$$

In comparing (24) with (12) we see that with a redefinition of the parameters, the iteration does not bring anything new so that the one-parameter family found above is really unique. With the relation (20) we also find that with an addition followed by a deletion of the ground state, then

$$\hat{v}^{(2)}=\log\phi_{2,0}(\lambda^{(l)},x)+\text{constant} \tag{25}$$

leading then to the same conclusion as above. For this case, however, it is also possible to initiate a mechanism of successive deletion and addition of higher excited states yielding an n-parameter family of isospectral potentials but with alteration of the phase shift, n being the number of excited states. This technique has already been developed by Keung *et al* (1989).

Remarking that iteration is in fact related to repeated operations of the $\hat{C}$ transformation $((\hat{C})^p)$ that is to say to a nonlinear transformation, another property which may be useful later can also be proved. In fact, after p transformations, the resulting superpotentials are, from (4):

$$\hat{v}^{(p)}_{\alpha,\beta}=v\pm p\log c(x).$$

The corresponding potentials $\hat{V}_\pm$ pertaining to the index α are, with the above method:

$$\hat{V}^{(p)}_{+}=V_+ \qquad \hat{V}^{(p)}_{-}=V_-+v'\left(\frac{c'}{c}\right)+[(p+1)^2-2]\left(\frac{c'}{c}\right)^2 \tag{26}$$

in which $c(x)$ is given by a relation similar to (12). For $p=l$, it can be shown that the result in equation (21*b*) is recovered exactly. This remark therefore opens new possibilities for constructing isospectral potentials. For example if we take a combination of powers of $\hat{C}$,

$$P=\sum_p a_p(\hat{C})^p \qquad a_p=\text{coefficients}$$

then the resulting superpotentials are simply:

$$\hat{v}'=av'+b(\log c(\lambda, x))$$

in which

$$a=\sum_p a_p \qquad b=\sum_p pa_p.$$

Some other consequences. It is instructive to digress on some other aspects of the $\hat{C}$ transformation which are not directly related to isospectral potentials but which may be useful in other problems. In the following we shall explore two simple cases as examples:

Case 1. In spite of the non-adjointness of the quantities $\hat{A}^{\pm}_{\alpha,\beta}$, we may always construct two following second-order differential equations:

$$\hat{A}^-_\beta\hat{A}^+_\alpha\bar{\bar{\phi}}_1=2E\bar{\bar{\phi}}_1 \qquad \hat{A}^{\pm}_\alpha\hat{A}^-_\beta\bar{\bar{\phi}}_2=2E\bar{\bar{\phi}}_2 \tag{27}$$

or more explicitly:

$$F(x)=-2\frac{\mathrm{d}}{\mathrm{d}x}(\log c)\frac{\mathrm{d}}{\mathrm{d}x}-\frac{1}{c}\frac{\mathrm{d}^2c}{\mathrm{d}x^2}$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_-+F(x)\right]\bar{\bar{\phi}}_1=2E\bar{\bar{\phi}}_1 \tag{28a}$$

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+V_++F(x)\right]\bar{\bar{\phi}}_2=2E\bar{\bar{\phi}}_2. \tag{28b}$$

If $V_\pm$ are assumed to be shape invariant, then the corresponding Schrödinger equation for ϕ_1, ϕ_2 can be solved exactly. This means that the $\hat{C}$ transformation has enabled us to generate a class of non-quadratic second-order differential equations ($c(x)$ being arbitrary) which is exactly solvable (because $\bar{\bar{\phi}}_1$, $\bar{\bar{\phi}}_2$ are related to ϕ_1, ϕ_2 through (4)) but which are not SUSY partners.

Note also that for each given eigenvalue ($E=E_n$) we have a couple ($\bar{\bar{\phi}}_{1,n}$, $\bar{\bar{\phi}}_{2,n}$) which are related by

$$\bar{\bar{\phi}}_{1,n}=\frac{1}{\sqrt{E_n}}\hat{A}^-_\beta\bar{\bar{\phi}}_{2,n}$$

and if the $\hat{C}$ transformation destroys the adjointness of the ladder operator $A^{\pm}$, it leaves its commutation invariant i.e.

$$[\hat{A}^-_\beta, \hat{A}^+_\alpha]=[A^-, A^+]=-2v''. \tag{29}$$

Generalizing this idea, we may define the 'charge operator' $\hat{Q}^{\pm}$ by

$$\hat{Q}^+_\alpha=\begin{pmatrix}0 & \hat{A}^+_\alpha\\ 0 & 0\end{pmatrix} \qquad \hat{Q}^-_\beta=\begin{pmatrix}0 & 0\\ \hat{A}^-_\beta & 0\end{pmatrix} \tag{30}$$

and construct the 'Hamiltonian' $\hat{H}$ as

$$\hat{H}=\{\hat{Q}_{\alpha}^{+},\hat{Q}_{\beta}^{-}\}=\begin{pmatrix}\hat{A}_{\alpha}^{+}\hat{A}_{\beta}^{-} & 0\\ 0 & \hat{A}_{\beta}^{-}\hat{A}_{\alpha}^{+}\end{pmatrix} \tag{31}$$

where [,] and {,} represent the symbols of commutation and anticommutation operations. It can be verified that nilpotency $((\hat{Q}_{\alpha}^{+})^2=(\hat{Q}_{\beta}^{-})^2=0)$ and commutativity with $\hat{H}$ are conserved $[\hat{Q}_{\alpha}^{+},\hat{H}]=[\hat{Q}_{\beta}^{-},\hat{H}]=0$.

For these reasons, we find it convenient to denote the couple $(\bar{\bar{\phi}}_1,\bar{\bar{\phi}}_2)$ as pseudo-SUSY partners in order to make distinction with the couples $(\bar{\bar{\phi}}_1,\bar{\bar{\phi}}_{1,s})$, $(\bar{\bar{\phi}}_2,\bar{\bar{\phi}}_{2,s})$ which are genuine SUSY partners.

Case 2. There is no loss of generality by considering the inverse of (4), i.e. let

$$\bar{\bar{\phi}}=C\phi \tag{32}$$

because it corresponds merely to an inversion of the indices $\alpha\rightleftarrows\beta$. We may set $c(x)=f^m(x)$, where m is a parameter and f the unknown function. Consider now the coordinate transformation $x\rightarrow r$ such that

$$f^{2m}=\frac{\mathrm{d}r}{\mathrm{d}x}.$$

Noting

$$\frac{\mathrm{d}}{\mathrm{d}x}=f^{2m}\frac{\mathrm{d}}{\mathrm{d}r}\qquad\text{and}\qquad\frac{\mathrm{d}}{\mathrm{d}x}(\log c)=mf^{2m}\frac{\mathrm{d}}{\mathrm{d}r}(\log f)$$

the two equations (27) can be written as:

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{1}{f^{4m}}\left\{m\frac{\mathrm{d}^2}{\mathrm{d}x^2}(\log f)-m^2\left[\frac{\mathrm{d}}{\mathrm{d}x}(\log f)\right]^2+V_{\mp}-E\right\}+\varepsilon_{\mp}\right]\bar{\bar{\phi}}_{1,2}=\varepsilon_{\mp}\bar{\bar{\phi}}_{1,2} \tag{33}$$

where the quantities $\varepsilon_{\mp}\bar{\bar{\phi}}_{1,2}$ result from the technique of 'adding terms' in order to have a conventional eigenvalue problem. It is interesting to note that if we take $m=\frac{1}{2}$, these equations, after some minor modifications in the notations (for example if n labels a given state, we have $E_{n+1}^{(1)}=E_n^{(2)}$, etc), reduce exactly to the form given in Cooper *et al* (1989) where the method is referred to as the 'f operator transformation'. It has been used to show the non-preservation of supersymmetry and shape invariance, for example, a shape-invariant potential such as the generalized Pösch–Teller or the oscillator potential are transformed into another type, the Natanzon potential which is not shape invariant but exactly solvable (Cooper *et al* 1987).

We may check the practical use of the $\hat{C}$ transformation on the simple example of the oscillator problem with the form already used in the above reference:

$$v'=\tfrac{1}{2}\omega x-b$$

ω, b being constants. The component ϕ_1 is $\phi f_1=N_nH_n(\sqrt{\omega}\bar{x})\,\mathrm{e}^{-\bar{x}^2/2}$ with $\bar{x}=x-2b/\omega$, N_n being a normalization constant.

If we choose for $f(x)$ the form $f=\mathrm{d}r/\mathrm{d}x=x$ then the solution of the first equation (involving V_- and ε_-) in (33) is given by

$$\bar{\bar{\phi}}_1=f_{(2)}^{2m}\phi_1.$$

Therefore $\bar{\bar{\phi}}_1=2^{m/2}N_nz^{m/2}H_n(\sqrt{\omega}\bar{x})\,\mathrm{e}^{-\omega\bar{x}/2}$ with $m=\frac{1}{2}$, $\bar{\bar{\phi}}_1$ become identical to the result obtained by these authors. In this sense, we may see the 'f operator transformation' as a special case of the $\hat{C}$ transformation combined with appropriate coordinate transformation f.

The $\hat{C}$ transformation approach is helpful in gaining a deeper insight on some interesting aspects of supersymmetric quantum mechanics. For example, the use of relation (7) lead us to a conclusion in agreement with other authors (Khare and Sukhatme 1988) while with relation (20), a close connection with the Darboux theorem can be established. It serves to clarify a number of questions raised in previous papers (Cao 1990a, b) for instance concerning isospectral potentials generated by odd superpotential in which the difficulty resulting from the divergence of the integral in (7) is avoided with the use of the second approach. Furthermore it leads to a more generalized concept of the SUSY partnership which may be useful for further developments. It also shows that the 'operator transformation' pointed out by Cooper *et al* (1989) can be understood in a more general context and, therefore, is susceptible to playing the guiding role in the search for the possible existence of other types of exactly solvable potentials, for example, when we use a functional form of $c = c[g(x)]$, $g(x)$ being an auxiliary function discussed in Levai (1989).

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